The critical manifold of the Lorentz–Dirac equation

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PACS.03.50.De
PACS.41.20.-q

Abstract. We investigate the solutions to the Lorentz–Dirac equation and show that its solution flow has a structure identical to the one of renormalization group flows in critical phenomena. The physical solutions of the Lorentz–Dirac equation lie on the critical surface. The critical surface is repelling, i.e. any slight deviation from it is amplified and as a result the solution runs away to infinity. On the other hand, Dirac's asymptotic condition (acceleration vanishes for long times) forces the solution to be on the critical manifold. The critical surface can be determined perturbatively. Thereby one obtains an effective second order equation, which we apply to various cases, in particular to the motion of an electron in a Penning trap.

The Lorentz–Dirac equation governs the motion of a classical charge in prescribed external electromagnetic fields and including radiation reaction, i.e. the loss of energy due to radiation. In standard relativistic notation it reads [1]

$$m\dot{v}^\mu = eF^{\mu\nu}(x)v_\nu + (e^2/6\pi c^3)[\ddot{v}^\mu - \frac{1}{c^2}\,\dot{v}^\lambda\dot{v}_\lambda v^\mu]\,. \tag{1}$$

Here m is the (experimental) rest mass of the particle with charge e. $x^\mu(s)$ is the world line and $v^\mu(s) = \dot{x}^\mu(s)$ the velocity of the charge parametrized in

its eigentime s. The particle is subject to time–independent external fields as given through the electromagnetic field tensor $F^{\mu\nu}$. The first term in (1) is the Lorentz force while the second term describes the radiation reaction.

One obvious issue is to understand how the Lorentz–Dirac equation is related to the Maxwell–Lorentz equations with a suitable ultraviolet cut–off. This problem was studied extensively by Abraham, Lorentz, and many others, cf. [1, 2] for a detailed account. In his famous paper [3], Dirac circumvented the issue through a somewhat delicate splitting of the fields generated by a point charge. The, to our knowledge most complete formal derivation of (1) has been worked out by Nodvik [4]. Some rigorous results are [5, 6, 7]. For the purpose of this letter we regard the Lorentz–Dirac equation as given.

As noted already by Dirac, Eq. (1) has runaway solutions which grow exponentially in time, simply because for $F^{\mu\nu} = 0$ and in the approximation of small velocities we have $m\dot{\mathbf{v}} = (e^2/6\pi c^3)\ddot{\mathbf{v}}$. Dirac [3], reemphasized by Haag [8], postulated that the physical solutions to (1) must satisfy the asymptotic condition $\lim\limits_{s\to\infty} \dot{v}^\mu(s) = 0$, which, as extra bonus, is a substitute for the missing initial condition $\ddot{\mathbf{x}}(0)$. The validity of the asymptotic condition has been tested only in explicit cases [1, 9, 10]. With a general external field tensor $F^{\mu\nu}$ the solution behavior of (1) might be complicated and should expected to be chaotic. Physical and unphysical solutions might be thoroughly mixed. Thus in principle, for given $\mathbf{x}(0), \dot{\mathbf{x}}(0)$, there could be many solutions satisfying the asymptotic condition. Which one to pick then? On a more practical level, one would like to have a reliable numerical scheme not hampered by the instability of physical solutions.

The purpose of this letter is to explain that the solution flow of the Lorentz–Dirac equation has a structure familiar from the renormalization group flows in critical phenomena. The physical solutions lie on the critical surface, which contains attractive fixed points whose location depends on $F^{\mu\nu}$. Slightly off the critical surface, the solution grows exponentially fast, so to speak it flows to the high, resp. low, temperature fixed point. Our observation has two important implications. (1) The critical manifold is actually a surface of the form $\ddot{\mathbf{x}} = h(\mathbf{x}, \dot{\mathbf{x}})$. Thus, for given initial conditions $\mathbf{x}(0), \dot{\mathbf{x}}(0)$, there is exactly one solution on the critical surface and, as to be shown, it satisfies the asymptotic condition. (2) There is an effective second order equation, given below, which governs the motion on the critical surface. Thus the initial value problem is restored and the equation can

esasily be solved numerically. We will demonstrate the predictive power of the second order equation by a few examples, still handled without numerical integration, the physically most relevant of which is the motion of an electron in a Penning trap [12]. (This system was pointed out to us by Wolfgang Schleich).

In all applications the radiation reaction is a small correction to the Lorentz force equation, which means that the radiation reaction term, the *highest* derivative in (1), carries a small prefactor. Differential equations of such a type have been studied extensively through singular (or geometric) perturbation theory [13, 14], which is closely connected to the theory of center manifolds. The application to the Lorentz-Dirac equation is a little bit messy and has been carried out in [7]. Rather than trying to summarize these results, we believe it to be more instructive to illustrate the basic features of the method by using a fictituous mathematical example.

Let us consider then the ordinary differential equation of the form

$$\dot{x} = g(x, y)\,, \quad \varepsilon\dot{y} = y - h(x)\,, \tag{2}$$

$x(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$. We want to understand the behavior of solutions for small ε. If we simply set $\varepsilon = 0$, the second equation reduces to $y = h(x)$ and therefore $\dot{x} = g(x, h(x))$. The ambient phase space has disappeared and the motion takes place only on the one–dimensional surface $\{y = h(x)\}$. On the other hand we can go over to the slow time scale $\tau, \tau = \varepsilon^{-1}t$. Denoting differentiation with respect to τ by $'$, (2) reads

$$x' = \varepsilon g(x, y)\,, \quad y' = y - h(x)\,. \tag{3}$$

Setting now $\varepsilon = 0$, yields $x' = 0$, i.e. $x(\tau) = x_0$ and $y' = y - h(x_0)$. Thus the surface $\{y = h(x)\}$ consists exclusively of repelling fixed points. If $y(0) \neq h(x_0)$, the solution grows exponentially. In this sense the surface $\{y = h(x)\}$ is critical. The main result of geometric singular perturbation theory is that for small ε the critical surface persists and is of the form $\{y = h_\varepsilon(x)\}$. On the critical surface the motion is governed by

$$\dot{x} = g(x, h_\varepsilon(x))\,. \tag{4}$$

If $x(0), y(0)$ are off the critical surface, the solution to (2) diverges exponentially with rate $1/\varepsilon$.

Of course, abstractly only the existence of h_ε is asserted. Its concrete form must be extracted from (2). Fortunately we are allowed to determine

h_ε perturbatively (which is not the case for individual solutions). We make the ansatz $y = h_\varepsilon(x) = h_0(x) + \varepsilon h_1(x) + \mathcal{O}(\varepsilon^2)$ and insert in (2). This results in $\varepsilon\dot{y} = \varepsilon h_0'(x)\dot{x} + \mathcal{O}(\varepsilon^2) = \varepsilon h_0'(x)g(x, h_0(x)) + \mathcal{O}(\varepsilon^2) = h_0(x) + \varepsilon h_1(x) - h(x) + \mathcal{O}(\varepsilon^2)$. Comparing orders of ε yields $h_0(x) = h(x)$, $h_1(x) = h_0'(x)g(x, h_0(x))$. Inserting in (4), we have the effective equation of motion

$$\dot{x} = g(x, h(x)) + \varepsilon\partial_y g(x, h(x))h'(x)g(x, h(x))\,, \tag{5}$$

valid up to an error of order ε^2.

Returning to the Lorentz–Dirac equation, by the same argument it has a repelling critical surface of the form $\{\ddot{\mathbf{x}} = h(\mathbf{x}, \dot{\mathbf{x}})\}$. To understand the motion on the critical surface we fix one inertial frame and denote the position and velocity three–vectors by $\mathbf{r}(t), \mathbf{u}(t)$. If ϕ is the external electrostatic potential, then the energy balance reads

$$\begin{aligned}\frac{d}{dt}&[\gamma mc^2 + e\phi(\mathbf{r}) - (e^2/6\pi c^3)\gamma^4(\mathbf{u}\cdot\dot{\mathbf{u}})] \\ &= -(e^2/6\pi c^3)[\gamma^4\dot{\mathbf{u}}^2 + c^{-2}\gamma^6(\mathbf{u}\cdot\dot{\mathbf{u}})^2]\end{aligned} \tag{6}$$

with $\gamma = 1/\sqrt{1 - \mathbf{u}^2/c^2}$. We integrate both sides of (6) in time. Since on the critical manifold the Schott term $-(e^2/6\pi c^3)\gamma^4(\mathbf{u}\cdot\dot{\mathbf{u}})$ is bounded, we conclude that

$$\int_0^\infty dt[\gamma^4\dot{\mathbf{u}}^2 + c^{-2}\gamma^6(\mathbf{u}\cdot\dot{\mathbf{u}})^2] < \infty \tag{7}$$

on the critical surface. This is possible only if the asymptotic condition $\lim\limits_{t\to\infty}\dot{\mathbf{u}}(t) = 0$ holds. Off the critical manifold $\dot{\mathbf{u}}(t)$ diverges. Thus given $\mathbf{r}(0), \dot{\mathbf{r}}(0)$, the asymptotic condition singles out the *unique* $\ddot{\mathbf{r}}(0)$ on the critical surface.

Inserting the asymptotic condition in (1), we see that $-\nabla\phi(\mathbf{r}(t)) \to 0$ as $t \to \infty$, which implies in essence two distinct scenarios. (i) The particle is scattered into a region where $F^{\nu\mu} = 0$. Then $\mathbf{u}(t)$ has a limit as $t \to \infty$ and $\mathbf{r}(t)$ grows linearly. (ii) The motion is bounded. Then the particle comes to rest, $\lim\limits_{t\to\infty}\mathbf{u}(t) = 0$, at a point where the electrostatic force, $-\nabla\phi$, vanishes. Note that in general the condition $-\nabla\phi(\mathbf{r}_\infty) = 0$ does not determine the asymptotic position $\mathbf{r}_\infty$. E.g. a uniform magnetic field is confining even for $\phi(\mathbf{r}) = 0$.

In analogy to (5), our next task is to derive an effective second order equation for the motion on the critical surface. We follow the steps leading to (5) and obtain

$$
\begin{aligned}
m\dot v^\mu &= eF^{\mu\nu}(\mathbf{x})v_\nu + (e^2/6\pi c^3)\{(e/m)v^\sigma(\partial_\sigma F^{\mu\nu}(\mathbf{x}))v_\nu \\
&+(e/m)^2 F^{\mu\alpha}(\mathbf{x})F_\alpha{}^\nu(\mathbf{x})v_\nu + (e/mc)^2 F^{\sigma\alpha}(\mathbf{x})F_\alpha{}^\nu(\mathbf{x})v_\sigma v_\nu v^\mu\}\,. \qquad (8)
\end{aligned}
$$

In principle one could compute also higher order terms. But they have the same magnitude as those contributions neglected already in the derivation of the Lorentz–Dirac equation. In addition (8) correctly describes the long time behavior as dominated by radiation reaction. Higher orders yield no qualitative change and, at best, make a minute correction of relative order 10^{-24} or even smaller in concrete examples.

Eq. (8) appears in the second volume of the course in theoretical physics by Landau and Lifshitz [11], who were guided by the insight that radiation reaction must have a small effect. One can only speculate why the Landau and Lifshitz equation (8) is apparently ignored in the literature. For sure, they do not discuss the structure of the flow with its critical manifold nor the relation to the asymptotic condition.

There are several cases of interest where the solution to (8) can still be handled analytically. The first one is a vanishing magnetic field and an electrostatic potential varying only along the 2–axis. Setting $\mathbf{r} = (0, y, 0)$, $\mathbf{u} = (0, \dot y, 0)$, $e\phi(\mathbf{r}) = V(y)$, the one–dimensional motion is governed by

$$
\frac{d}{dt}(m\gamma\dot y) = -V'(y) - (e^2/6\pi c^3)(1/m)V''(y)\gamma\dot y\,. \qquad (9)
$$

If V is convex, the energy is damped monotonically. The particular case of a quadratic potential is studied in the recent third edition of the textbook by Jackson [15] in the context of line breadth and level shift of a radiating oscillator. However, if V is periodic, say $V(y) = V_0\cos(k_0 y)$, then at the maxima the particle gains in energy from the near field, a process dominated by the energy loss at the minima. For long times the particle comes to rest. Also if V has a linear piece, then in this spatial interval the charge is accelerated without friction.

An experimentally more accessible set–up is the motion in a uniform magnetic field $(0, 0, B)$. Then (8) simplifies to

$$
\frac{d}{dt}(m\gamma\mathbf{u}) = eB\mathbf{u}^\perp - (e^2/6\pi c^3)(eB/m)^2\gamma^2\mathbf{u} \qquad (10)
$$

with $\mathbf{u} = (u_1, u_2, 0) = u(\cos\varphi, \sin\varphi, 0)$, $\mathbf{u}^\perp = (-u_2, u_1, 0)$. Setting $\alpha = e^2/6\pi mc^3$ and the cyclotron frequency $\omega_c = eB/m$, one obtains $du/d\varphi = -\alpha\omega_c\varphi$, i.e. $u(\varphi) = u(0)\exp[-\alpha\omega_c\varphi]$. For electrons $\alpha\omega_c = 8.8\times10^{-18}B[\text{Gauss}]$, which shows that radiative reaction is very small even at high fields. The charge spirals to its central rest point. Using (10) the radius shrinks as

$$r(t) = r_0 e^{-\alpha\omega_c^2 t}/(1 + (\gamma - 1)((1 - e^{-2\alpha\omega_c^2 t})/2))\,. \tag{11}$$

In the ultrarelativistic regime, $\gamma \gg 1$, (11) simplifies to $r(t) = r_0(1 + \gamma\alpha\omega_c^2 t)^{-1}$, provided $\alpha\omega_c^2 t \ll 1$. To have some order of magnitude, in the case of an electron, $\alpha\omega_c^2 = 1.6 \times 10^{-6}(B[\text{Gauss}])^2/\sec$ and $r_0 = 1.7 \times 10^{-3}(\gamma/(B[\text{Gauss}]))\,\text{m}$. Thus for $B = 10^3\,\text{Gauss}$ and an ultrarelativistic $\gamma = 6 \times 10^4$ the radius shrinks within $0.9\,\sec$ from its initial value $r_0 = 10\,\text{cm}$ to $r = 1\,\mu\text{m}$, at which time the electron has made 2×10^{14} revolutions. Only then the power law, $(1 + t)^{-1}$, crosses over to an exponential damping.

Our third example is the motion of an electron in a Penning trap [12]. The electron is subject to a uniform magnetic field, as before, and in addition to the electrostatic quadrupol potential

$$e\phi(x, y, z) = \frac{1}{2}\, m\omega_z^2(-\frac{1}{2}x^2 - \frac{1}{2}y^2 + z^2)\,. \tag{12}$$

A non–relativistic approximation suffices and in (8) we only keep terms to linear order. Then the in–plane and axial motion decouple and satisfy, with $\mathbf{r} = (x, y), \mathbf{u} = (u_1, u_2)$,

$$\begin{aligned} \ddot{\mathbf{r}} &= \frac{1}{2}\,\omega_z^2\mathbf{r} + \omega_c\mathbf{u}^\perp - \frac{e^2}{6\pi c^3 m}\,\{(\omega_c^2 - \frac{1}{2}\,\omega_z^2)\mathbf{u} + \frac{1}{2}\omega_c\omega_z^2\mathbf{r}^\perp\}\,, && (13)\\ \ddot{z} &= -\omega_z^2 z - \frac{e^2}{6\pi c^3 m}\,\omega_z^2\dot{z}\,. && (14) \end{aligned}$$

The in–plane motion can be solved easily. Without friction there are two modes with frequencies $\omega_\pm = \frac{1}{2}\,(\omega_c \pm (\omega_c^2 - 2\omega_z^2)^{1/2})$. For the damping coefficients first order perturbation theory is sufficiently accurate with the result

$$\gamma_+ = \frac{e^2}{6\pi c^3 m}\,\frac{\omega_+^3}{\omega_+ - \omega_-}, \quad \gamma_- = \frac{e^2}{6\pi c^3 m}\,\frac{\omega_-^3}{\omega_- - \omega_+}\,, \tag{15}$$

in agreement with a QED resonance computation [12]. Of course with some extra effort, one could handle also nonlinear and relativistic effects. We

emphasize that (15) is beyond the capability of Larmor's formula, which works only for a single mode.

Note that $\omega_- < \omega_+$ and therefore $\gamma_- < 0$ which reflects that a loss of energy due to radiation lowers the potential energy and increases the orbit size. In practice $\omega_c/2\pi = 164\,\mathrm{GHz}$ and $\omega_z/2\pi = 62\,\mathrm{MHz}$. Inserting in (15) leads to $\gamma_+^{-1} = 8 \times 10^{-2}$ sec and $\gamma_-^{-1} = -3 \times 10^{14}$ sec. Experimentally, one observes that the magnetron motion (ω_-) is stable over weeks, whereas the cyclotron motion (ω_+) decays to equilibrium within fractions of a second.

To summarize, we have investigated the solution flow of the Lorentz–Dirac equation and discovered that in its structure it is identical to renormalization group flows in critical phenomena. The physical solutions are on the critical manifold and are governed there by an effective second order equation. This equation is not plagued by the difficulties usually associated with the Lorentz–Dirac equation. In particular, the solutions to (8) are stable and have the correct long–time behavior. Our examples show how radiation damping can be handled systematically and with ease. It would be of interest to have more stringent experimental tests. E.g. one could decrease the magnetic field in the Penning trap, so that the two modes mix better, and try to reach the resonance point $\omega_c^2 = 2\omega_z^2$, where the life–times should vanish according to (15). A further interesting possibility is to turn the magnetic field out of the symmetry axis. Then all three modes mix resulting in damping coefficients which can be understood only on the basis of (8).

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I am most grateful to Fritz Rohrlich for instructive discussions and for the hint that the independently derived Eq. (8) appeared in Landau and Lifshitz already a long time ago. I am indebted to Wolfgang Schleich for pointing out the Penning trap as an interesting application. I thank M. Rauscher for details on synchroton sources.

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